

# Tangent cones to generalised theta divisors and generic injectivity of the theta map

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## Abstract

Let  $C$  be a Petri general curve of genus  $g$  and  $E$  a general stable vector bundle of rank  $r$  and slope  $g - 1$  over  $C$  with  $h^0(C, E) = r + 1$ . For  $g > (r + 1)^2$ , we show how the bundle  $E$  can be recovered from the tangent cone to the theta divisor  $\Theta_E$  at  $\mathcal{O}_C$ . We use this to give a constructive proof and a sharpening of Brivio and Verra's theorem that the theta map  $SU_C(r) \dashrightarrow |r\Theta|$  is generically injective for large values of  $g$ .

## 1 Introduction

Let  $C$  be a nonhyperelliptic curve of genus  $g$  and  $L \in \text{Pic}^{g-1}(C)$  a line bundle with  $h^0(C, L) = 2$  corresponding to a general double point of the Riemann theta divisor  $\Theta$ . It is well known that the projectivised tangent cone to  $\Theta$  at  $L$  is a quadric hypersurface  $R_L$  of rank 4 in the canonical space  $|K_C|^*$ , which contains the canonically embedded curve.

Quadrics arising from tangent cones in this way have been much studied: Green [Gre84] showed that the  $R_L$  span the space of all quadrics in  $|K_C|^*$  containing  $C$ ; and both Kempf and Schreyer [KS88] and Ciliberto and Sernesi [CS92] have used the quadrics  $R_L$  in various ways to give new proofs of Torelli's theorem.

In another direction: Via the Riemann–Kempf singularity theorem [Kem73], one sees that the rulings on  $R_L$  cut out the linear series  $|L|$  and  $|K_C L^{-1}|$  on the canonical curve. Thus the data of the tangent cone and the canonical curve allows one to reconstruct the line bundle  $L$ . In this article we study a related construction for vector bundles of higher rank.

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Let  $V \rightarrow C$  be a semistable vector bundle of rank  $r$  and integral slope  $h$ . We consider the set

$$\left\{ M \in \text{Pic}^{g-1-h}(C) : h^0(C, V \otimes M) \geq 1 \right\}. \quad (1)$$

It is by now well known that for general  $V$ , this is the support of a divisor  $\Theta_V$  algebraically equivalent to a translate of  $r \cdot \Theta$ . If  $V$  has trivial determinant, then in fact  $\Theta_V$ , when it exists, belongs to  $|r\Theta|$ .

For general  $V$ , the projectivised tangent cone  $\mathcal{T}_M(\Theta_V)$  to  $\Theta_V$  at a point  $M$  of multiplicity  $r+1$  is a determinantal hypersurface of degree  $r+1$  in  $|K_C|^*$  (see for example Casalaina Martin–Teixidor i Bigas [CMTiB11]). Our first result (§4.2) is a construction which from  $\mathcal{T}_M(\Theta_V)$  recovers the bundle  $V \otimes M$ , up to the involution  $V \otimes M \mapsto K_C \otimes M^{-1} \otimes V^*$ . This is valid for  $g \geq (r+1)^2$  and for general  $C$  and  $V$ .

We apply this construction to give an improvement of a result of Brivio and Verra [BV12]. To describe this application, we need to recall some more objects. Write  $SU_C(r)$  for the moduli space of semistable bundles of rank  $r$  and trivial determinant over  $C$ . The association  $V \mapsto \Theta_V$  defines a map

$$\mathcal{D}: SU_C(r) \dashrightarrow |r\Theta| = \mathbb{P}^{r^g-1}, \quad (2)$$

called the *theta map*. Drezet and Narasimhan [DN89] showed that the line bundle associated to the theta map is the ample generator of the Picard group of  $SU_C(r)$ . Moreover, the indeterminacy locus of  $\mathcal{D}$  consists of those bundles  $V \in SU_C(r)$  for which (1) is the whole Picard variety. This has been much studied; see for example Pauly [Paul0], Popa [Pop99] and Raynaud [Ray82].

Brivio and Verra [BV12] showed that  $\mathcal{D}$  is generically injective for a general curve of genus  $g \geq \binom{3r}{r} - 2r - 1$ , partially answering a conjecture of Beauville [Bea06, §6]. We apply the aforementioned construction to give the following sharpening of Brivio and Verra's result:

**Theorem 1.1.** *For  $r \geq 3$  and  $C$  a Petri general curve of genus  $g > (r+1)^2$ , the theta map (2) is generically injective.*

In addition to giving the statement for several new values of  $g$  (our lower bound for  $g$  depends quadratically on  $r$  rather than exponentially), our proof is constructive, based on the method mentioned above for explicitly recovering the bundle  $V$  from the tangent cone to the theta divisor at a point of multiplicity  $r+1$ . This gives a new example, in the context of vector bundles, of the principle apparent in [KS88] and [CS92] that the geometry of a theta divisor at a sufficiently singular point can encode essentially all the information of the bundle and/or the curve.

Our method works also for  $r=2$  if  $g \geq 11$ , but in this case much more is already known: Narasimhan and Ramanan [NR69] showed, for  $g=2$  and  $r=2$ , that  $\mathcal{D}$  is an isomorphism  $SU_C(2) \xrightarrow{\sim} \mathbb{P}^3$ , and van Geemen and Izadi [vGI01] generalised this

statement to nonhyperelliptic curves of higher genus. Note that our proof of Theorem 1.1 is not valid for hyperelliptic curves. See Remark 5.3.

Here is a more detailed summary of the article. In §2, we study semistable bundles  $E$  of slope  $g - 1$  for which the *Petri trace map*

$$\bar{\mu}: H^0(C, E) \otimes H^0(C, K_C \otimes E^*) \rightarrow H^0(C, K_C)$$

is injective. A bundle  $E$  with this property will be called *Petri trace injective*. We construct such bundles over a general curve  $C$ . We prove (Proposition 2.6) that for general  $V \in SU_C(r)$  and for  $n^2 \leq g$ , the locus of points of multiplicity  $n$  in  $\Theta_V$  contains a component of the expected dimension  $g - n^2$ . In §3, we give a generality condition on  $C$  (weaker than the Petri property) which is sufficient to imply the statements in §2. This partially generalises a result of Brivio [Br15].

Suppose now that  $E$  is a vector bundle of slope  $g - 1$  with  $h^0(C, E) \geq 1$ . If  $\Theta_E$  is defined and  $\text{mult}_{\mathcal{O}_C}(\Theta_E) = h^0(C, E)$ , then the tangent cone to  $\Theta_E$  at  $\mathcal{O}_C$  is a determinantal hypersurface in  $|K_C|^* = \mathbb{P}^{g-1}$  containing the canonical embedding of  $C$ . We prove (Proposition 4.3 and Corollary 4.5) that if  $C$  is a general curve of genus  $g \geq (r + 1)^2$ , and  $E$  a Petri trace injective bundle of rank  $r$  and slope  $g - 1$  with  $h^0(C, E) = r + 1$ , then the bundle  $E$  can be reconstructed up to the involution  $E \mapsto K_C \otimes E^*$  from a certain determinantal representation of the tangent cone to  $\Theta_E$  at  $\mathcal{O}_C$ . By a classical result of Frobenius (whose proof we sketch in Proposition 4.7), any two such representations are equivalent up to transpose. The generic injectivity of the theta map for a Petri general curve (Theorem 5.1) can then be deduced by combining these facts and the statement in §2 that the theta divisor of a general  $V \in SU_C(r)$  contains a suitable point of multiplicity  $r + 1$ .

We assume throughout that the ground field is  $\mathbb{C}$ . The reconstruction of  $E$  from its tangent cone in §4.2 is valid for an algebraically closed field of characteristic zero or  $p > 0$  not dividing  $r + 1$ .

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## 2 Theta divisors of semistable vector bundles

### 2.1 Theta divisors and Petri trace injectivity

Let  $C$  be a projective smooth curve of genus  $g \geq 2$ . Let  $V \rightarrow C$  be a stable vector bundle of rank  $r \geq 2$  and integral slope  $h$ , and consider the locus

$$\left\{ M \in \text{Pic}^{g-1-h}(C) : h^0(C, V \otimes M) \geq 1 \right\}. \quad (3)$$

If this is not the whole of  $\text{Pic}^{g-1-h}(C)$ , then it is the support of the theta divisor  $\Theta_V$ .

The theta divisor of a vector bundle is a special case of a *twisted Brill–Noether locus*

$$B_{1,g-1-h}^n(V) := \left\{ M \in \text{Pic}^{g-1-h}(C) : h^0(C, V \otimes M) \geq n \right\}. \quad (4)$$

The following is central in the study of these loci (see for example Teixidor i Bigas [TiB14, §1]): For  $E \rightarrow C$  a vector bundle, we consider the *Petri trace map*:

$$\bar{\mu} : H^0(C, E) \otimes H^0(C, K_C \otimes E^*) \xrightarrow{\mu} H^0(C, K_C \otimes \text{End}(E)) \xrightarrow{\text{tr}} H^0(C, K_C). \quad (5)$$

Then for  $E = V \otimes M$  and  $M \in B_{1,g-1-h}^n(V) \setminus B_{1,g-1-h}^{n+1}(V)$ , the Zariski tangent space to  $B_{1,g-1-h}^n(V)$  at  $M$  is exactly  $\text{Im}(\bar{\mu})^\perp$ . This motivates a definition:

**Definition 2.1.** Suppose  $E \rightarrow C$  is a vector bundle with  $h^0(C, E) = n \geq 1$ . If the map  $\mu$  above is injective, we will say that  $E$  is *Petri injective*. If the composed map  $\bar{\mu}$  is injective, we will say that  $E$  is *Petri trace injective*.

**Remark 2.2.**

- (1) Clearly, a Petri trace injective bundle is Petri injective. For line bundles, the two notions coincide.
- (2) Suppose  $V \in U_C(r, d)$  where  $U_C(r, d)$  is the moduli space of semistable rank  $r$  vector bundles of degree  $d$ . If  $E = V \otimes M$  is Petri trace injective for  $M \in \text{Pic}^e(C)$ , then  $B_{1,e}^n(V)$  is smooth at  $M$  and of the expected dimension

$$h^1(C, \mathcal{O}_C) - h^0(C, V \otimes M) \cdot h^1(C, V \otimes M).$$

- (3) We will also use the usual generalised Brill–Noether locus

$$B_{r,d}^n = \{ F \in U_C(r, d) : h^0(C, F) \geq n \}.$$

If  $E$  is Petri injective then this is smooth and of the expected dimension

$$h^1(C, \text{End} E) - h^0(C, E) \cdot h^1(C, E)$$

at  $E$ . See for example Grzegorczyk and Teixidor i Bigas [GTiB09, §2].

- (4) Clearly, Petri injectivity and Petri trace injectivity are open conditions on families of bundles  $\mathcal{E} \rightarrow C \times B$  with  $h^0(C, \mathcal{E}_b)$  constant.

We will also need the notion of a Petri general curve:

**Definition 2.3.** A curve  $C$  is called *Petri general* if every line bundle on  $C$  is Petri injective.

By [Gie82], the locus of curves which are not Petri general is a proper subset of the moduli space  $M_g$  of curves of genus  $g$ , the so called *Gieseker–Petri locus*. The hyperelliptic locus is contained in the Gieseker–Petri locus. Apart from this, in general not much is known about the components of the Gieseker–Petri locus and their dimensions. For an overview of known results, we refer to [TiB88], [Far05] and [BS11] and the references cited there.

**Proposition 2.4.** *Suppose  $V$  is a stable bundle of rank  $r$  and integral slope  $h$ . Suppose  $M_0 \in \text{Pic}^{g-1-h}(C)$  satisfies  $h^0(C, V \otimes M_0) \geq 1$ , and furthermore that  $V \otimes M_0$  is Petri trace injective. Then the theta divisor  $\Theta_V \subset \text{Pic}^{g-1-h}(C)$  is defined. Furthermore, we have equality  $\text{mult}_{M_0} \Theta_V = h^0(C, V \otimes M_0)$ .*

*Proof.* Write  $E := V \otimes M_0$ . It is well known that via Serre duality,  $\bar{\mu}$  is dual to the cup product map

$$\cup: H^1(C, \mathcal{O}_C) \rightarrow \text{Hom}(H^0(C, E), H^1(C, E)).$$

By hypothesis, therefore,  $\cup$  is surjective. Since  $E$  has Euler characteristic zero,  $h^0(C, E) = h^1(C, E)$ . Hence there exists  $b \in H^1(C, \mathcal{O}_C)$  such that  $\cdot \cup b: H^0(C, E) \rightarrow H^1(C, E)$  is injective. The tangent vector  $b$  induces a deformation of  $M_0$  and hence of  $E$ , which does not preserve any nonzero section of  $E$ . Therefore, the locus

$$\{M \in \text{Pic}^{g-1-h}(C) : h^0(C, V \otimes M) \geq 1\}$$

is a proper sublocus of  $\text{Pic}^{g-1-h}(C)$ , so  $\Theta_V$  is defined. Now we can apply Casalaina–Martin and Teixidor i Bigas [CMTiB11, Proposition 4.1], to obtain the desired equality  $\text{mult}_{M_0} \Theta_V = h^0(C, V \otimes M_0)$ .  $\square$

## 2.2 Existence of sufficiently singular points

**Lemma 2.5.** *Suppose  $C$  admits a Petri injective line bundle  $N$  of degree  $g-1$  with  $h^0(C, N) = n$ . Suppose that  $r \geq 2$  and that*

$$g \geq \begin{cases} n^2 & : r \geq 3 \\ n^2 + 2 & : r = 2. \end{cases}$$

*Then there exist stable bundles  $E \in U_C(r, r(g-1))$  with  $h^0(C, E) = n$  and which are Petri trace injective.*

*Proof.* Let  $F$  be a general stable bundle of rank  $r - 1$  and degree  $(r - 1)(g - 1)$ , and let

$$0 \rightarrow N \xrightarrow{\iota} E_0 \rightarrow F \rightarrow 0$$

be a nontrivial extension. By Narasimhan and Ramanan [NR69, Lemma 4.1], the bundle  $E_0$  is semistable and simple. Since  $F$  is general,  $h^0(C, F) = h^1(C, F) = 0$ . Hence the inclusion  $\iota: N \rightarrow E_0$  induces isomorphisms

$$\iota: H^0(C, N) \xrightarrow{\sim} H^0(C, E_0) \quad \text{and} \quad {}^t\iota: H^0(C, K_C \otimes E_0^*) \xrightarrow{\sim} H^0(C, K_C \otimes N^{-1}). \quad (6)$$

In particular,  $h^0(C, E_0) = h^0(C, K_C \otimes E_0^*) = n$ .

We claim that  $E_0$  is Petri trace injective. Let  $\sigma_1, \dots, \sigma_k$  and  $\tau_1, \dots, \tau_k$  be bases of  $H^0(C, E_0)$  and  $H^0(C, K_C \otimes E_0^*)$  respectively. Write  $\widetilde{\sigma}_i$  and  $\widetilde{\tau}_j$  for the preimages of  $\sigma_i$  and  $\tau_j$  via (6). For each  $i, j$  we have a commutative diagram

$$\begin{array}{ccccc} E_0 & \xrightarrow{\tau_j} & K_C & \xrightarrow{\text{Id} \otimes \sigma_i} & K_C \otimes E_0 \\ \uparrow \iota & \nearrow \widetilde{\tau}_j & & \searrow \text{Id} \otimes \widetilde{\sigma}_i & \uparrow \text{Id} \otimes \iota \\ N & & & & K_C \otimes N \end{array}$$

where the top row defines the twisted endomorphism

$$\bar{\mu}(\sigma_i \otimes \tau_j) \in H^0(C, K_C \otimes \text{End} E_0).$$

Clearly this has rank one. As it factorises via  $K_C \otimes N$ , at a general point of  $C$ , the eigenspace corresponding to the single nonzero eigenvalue is the fibre of  $N \subset E_0$ . Hence the trace of  $\bar{\mu}(\sigma_i \otimes \tau_j)$  may be identified with the restriction to  $N$ . By the diagram, we may identify this trace with

$$\widetilde{\sigma}_i \cdot \widetilde{\tau}_j \in H^0(C, K_C).$$

Since by hypothesis the multiplication map  $H^0(C, N) \otimes H^0(C, K_C N^{-1}) \rightarrow H^0(C, K_C)$  is injective, the elements  $\bar{\mu}(\sigma_i \otimes \tau_j)$  are independent in  $H^0(C, K_C)$ . Hence  $E_0$  is Petri trace injective.

We now show that  $E_0$  can be deformed to a stable bundle preserving the property  $h^0(C, E_0) = n$ . As  $E_0$  is simple, first order infinitesimal deformations of  $E_0$  are injectively parametrised by  $H^1(C, \text{End} E_0)$ . The subspace of deformations preserving all global sections is given by  $T_{E_0} B_{r, r(g-1)}^n = \text{Im}(\mu)^\perp$ , which by the previous paragraph is of codimension  $n^2$ . On the other hand, as  $N$  is the only destabilising subbundle of  $E_0$ , the component of the strictly semistable locus containing  $E_0$  consists of extensions  $0 \rightarrow N' \rightarrow E' \rightarrow F' \rightarrow 0$  where  $N' \in \text{Pic}^{g-1}(C)$  and  $F' \in U_C(r-1, (r-1)(g-1))$ . The locus of such bundles has dimension at most equal to

$$\dim \text{Pic}^{g-1}(C) + \dim U_C(r-1, (r-1)(g-1)) + h^1(C, \text{Hom}(F, N)) - 1. \quad (7)$$

As  $F$  is stable,  $h^0(C, \text{Hom}(F, N)) = 0$ . Hence

$$h^1(C, \text{Hom}(F, N)) = -\chi(C, \text{Hom}(F, N)) = (r-1)(g-1).$$

Hence (7) is at most equal to

$$\begin{aligned} g + (r-1)^2(g-1) + 1 + (r-1)(g-1) - 1 &= (g-1)(r^2 - r + 1) + 1 \\ &= \dim U_C(r, r(g-1)) - (r-1)(g-1). \end{aligned}$$

Our hypotheses on  $r$ ,  $g$  and  $n$  imply that  $(r-1)(g-1) > n^2$ . Hence the component of the strictly semistable locus containing  $E_0$  has greater codimension than the locus of bundles with  $n$  independent sections. Thus a general deformation  $E$  of  $E_0$  preserving the property  $h^0(C, E) = n$  is a stable vector bundle. As Petri trace injectivity is an open condition on families of bundles with  $n$  sections, we may assume that  $E$  is Petri trace injective.  $\square$

**Proposition 2.6.** *Suppose  $C$  admits a Petri injective line bundle  $N$  of degree  $g-1$  with  $h^0(C, N) = n$ . Then for general  $V \in SU_C(r)$ , the theta divisor  $\Theta_V \in |r\Theta|$  exists and contains at least one point  $M$  of multiplicity  $n$  such that  $V \otimes M$  is Petri trace injective.*

*Proof.* By Lemma 2.5, we may choose a stable, Petri trace injective vector bundle  $E_1 \in U_C(r, r(g-1))$  with  $h^0(C, E_1) = n$ . By Remark 2.2 (1), (3) and (4), there is an open (not necessarily dense) subset  $\tilde{B}$  of  $B_{r, r(g-1)}^n$  which is smooth and of the expected dimension at  $E_1$  and in which a generic bundle is stable, satisfies  $h^0(C, E) = n$  and is Petri trace injective.

Consider now the map  $a: SU_C(r) \times \text{Pic}^{g-1}(C) \rightarrow U_C(r, r(g-1))$  given by  $(V, M) \mapsto V \otimes M$ . This is an étale cover of degree  $r^{2g}$ . We write  $\tilde{P}$  for the inverse image  $a^{-1}(\tilde{B})$ . Set-theoretically, we have

$$\begin{aligned} \tilde{P} &\subseteq \{(V, M) \in SU_C(r) \times \text{Pic}^{g-1}(C) : \\ &\quad h^0(C, V \otimes M) = n \text{ and } V \otimes M \text{ is stable and Petri trace injective}\}. \end{aligned}$$

Since  $a$  is étale, we have  $T_{(V, M)}\tilde{P} \cong T_{V \otimes M}\tilde{B}$  for each  $(V, M) \in \tilde{B}$ . In particular,

$$\dim \tilde{P} = \dim \tilde{B} = \dim U_C(r, r(g-1)) - n^2. \quad (8)$$

We write  $p$  for the projection  $SU_C(r) \times \text{Pic}^{g-1}(C) \rightarrow SU_C(r)$ , and  $p_n$  for the restriction

$$p|_{\tilde{P}}: \tilde{P} \rightarrow SU_C(r).$$

**Claim:**  $p_n$  is dominant.

To see this: For  $(V, M) \in \tilde{P}$ , the locus  $p_n^{-1}(V)$  is identified with an open subset of the twisted Brill-Noether locus

$$B_{1, g-1}^n(V) = \{M \in \text{Pic}^{g-1}(C) : h^0(C, V \otimes M) \geq n\} \subseteq \text{Pic}^{g-1}(C).$$

Moreover, for each such  $(V, M)$ , we have

$$\dim_M(p_n^{-1}(V)) = \dim\left(T_M\left(B_{1,g-1}^n(V)\right)\right) = \dim \operatorname{Im}(\bar{\mu})^\perp.$$

Since  $V \otimes M$  is Petri trace injective, this dimension is  $g - n^2$ . By semicontinuity, a general fibre of  $p_n$  has dimension at most  $g - n^2$ . Therefore, in view of (8), the image of  $p_n$  has dimension at least

$$(\dim U_C(r, r(g-1)) - n^2) - (g - n^2) = \dim(U_C(r, r(g-1))) - g = \dim(SU_C(r)).$$

As  $SU_C(r)$  is irreducible,  $p_n$  is dominant.

Now we can finish the proof: Let  $V \in SU_C(r)$  be general. By the claim, we can find  $(V, M) \in \tilde{P}$  such that  $h^0(C, V \otimes M) = n$  and  $V \otimes M$  is Petri trace injective. By Proposition 2.4, the theta divisor  $\Theta_V$  exists and satisfies  $\operatorname{mult}_M \Theta_V = h^0(C, V \otimes M) = n$ .  $\square$

**Remark 2.7.** This proof shows that for general  $V$ , the locus of points of multiplicity  $n$  in  $\Theta_V$  contains a component of dimension  $g - n^2$ , of which a general point  $M$  satisfies  $h^0(C, V \otimes M) = n$ . This gives a partial generalisation of Brivio [Bril5, Theorem 1.1 (1)].

**Remark 2.8.** By [TiBl4, Theorem 1.1], for generic  $E$  and  $C$ , Petri trace injectivity is known for all twists  $M \otimes E$  where  $M$  is a line bundle (not necessarily of degree  $g - 1$ ). However, if  $n \geq 2$ , the condition  $h^0(C, E) = n$  is a condition of codimension at least 4 on  $E \in U_C(r, r(g-1))$ , so we cannot immediately apply this result.

### 3 Curves with a Petri injective line bundle of degree $g - 1$ and $n$ global sections

**Notation:** In this section, we will consider the Brill-Noether loci  $B_{1,g-1}^n$  (classically denoted  $W_{g-1}^{n-1}(C)$ ) where the curve  $C$  may vary. Abusing notation, we will continue to write  $B_{1,g-1}^n$ , as the curve  $C$  should always be clear from the context.

Fix a positive integer  $n \geq 2$  and let  $C$  be a smooth curve of genus  $g \geq n^2$ . In order to perform the construction in §2.2, we are interested in curves with a Petri injective line bundle of degree  $g - 1$  with  $n$  sections. By Brill-Noether theory [ACGH85], the locus  $B_{1,g-1}^n$  is nonempty, since  $\rho = \rho(g, g - 1, n - 1) = g - n^2 \geq 0$ . Thus the Petri injectivity condition is equivalent to the question whether a reduced component of expected dimension  $\rho$  of  $B_{1,g-1}^n$  exists. Hence, we would like to characterise the complement of the following locus in the moduli space of genus  $g$  curves

$$X_n := \{C \in M_g : \text{every component of } B_{1,g-1}^n \setminus B_{1,g-1}^{n+1} \text{ is either} \\ \text{of dimension larger than expected or nonreduced}\}.$$



**Remark 3.1.** Note that  $X_n$  is contained in the Gieseker-Petri locus. For  $n = 2$ , it is well known that  $X_2$  coincides with the hyperelliptic locus in  $M_g$ . Hence, there always exists a Petri injective pencil of degree  $g - 1$  on nonhyperelliptic curves.

Here we will give a partial characterisation of  $X_n$ . Recall that the curve  $C$  is called *k-gonal* if the lowest degree of a nonconstant rational map  $C \rightarrow \mathbb{P}^1$  is  $k$ .

**Proposition 3.2.** *Fix a positive  $n \geq 2$ . Let  $k \geq n + 1$  and let  $C$  be a general  $k$ -gonal curve of genus  $g \geq n^2$ . Then there exists a Petri injective line bundle of degree  $g - 1$  with  $n$  global sections on  $C$ . In other words, the locus  $X_n$  does not completely contain the loci  $M_{g,k}^1 := \{C \in M_g : C \text{ has a } g_k^1\}$  for  $k \geq n + 1$ .*

In the case of an  $(n + 1)$ -gonal curve, we can apply a result of Coppens and Martens. We only state their result for the case that we will need it. The locus  $B_{1,d}^1$  is the locus of effective line bundles of degree  $d$  in  $\text{Pic}^d(C)$ .

**Lemma 3.3** ([CM02, Corollary p. 32] or [CM99, Proposition 2.3.1]). *Let  $C$  be a general  $(n + 1)$ -gonal curve of genus  $g \geq n^2$ . Then,  $(n - 1) \cdot g_{n+1}^1 + B_{1,g-n^2}^1$  and by duality*

$$K_C - \left( (n - 1) \cdot g_{n+1}^1 + B_{1,g-n^2}^1 \right)$$

*are irreducible components of  $B_{1,g-1}^n$  isomorphic to  $B_{1,g-n^2}^1$ . In particular, they are generically reduced and of the expected dimension  $\rho$ .*

To prove Proposition 3.2, we use the gonality stratification of the moduli space  $M_g$  of curves of genus  $g$ , a similar argument to that in [CM99]. This is a stratification of irreducible varieties

$$M_g = M_{g, \left\lceil \frac{g+2}{2} \right\rceil}^1 \supset \cdots \supset M_{g, n+1}^1 \supset \cdots \supset M_{g, 2}^1$$

(see [Far01]). Note that the gonality of a general curve is  $\left\lceil \frac{g+2}{2} \right\rceil$  and that our hypotheses imply  $n + 1 \leq \left\lceil \frac{g+2}{2} \right\rceil$ , with equality if and only if  $n = 2$  and  $g = 4$ .

Let  $U \subset M_g$  be the open subset of the moduli space consisting of curves with a Petri injective line bundle  $L \in B_{1,g-1}^n \setminus B_{1,g-1}^{n+1}$ . Note that  $U$  contains the open and dense locus of Petri general curves. Using Lemma 3.3, we see that the intersection of  $U$  and  $M_{g, n+1}^1$  is non-empty. Hence,  $U$  intersects all  $M_{g, k}^1$  for  $k \geq n + 1$ . We conclude that  $B_{1,g-1}^n$  has an irreducible generically reduced component of expected dimension for a general curve of gonality  $k \geq n + 1$ ; equivalently, such a curve admits a Petri injective line bundle of degree  $g - 1$  with  $n$  global sections.

**Remark 3.4.** It seems reasonable to conjecture that Proposition 3.2 is in a sense optimal for  $n > 2$ . For  $n = 3$ : Maroni [Mar46] classified the Brill-Noether loci for trigonal curves, showing that the statement is false in this case. See also Coppens-Martens [CM99], [CM00] and Park [Par02].

## 4 Reconstruction of bundles from tangent cones to theta divisors

### 4.1 Tangent cones

Let  $Y$  be a normal variety and  $Z \subset Y$  a divisor. Let  $p$  be a smooth point of  $Y$  which is a point of multiplicity  $n \geq 1$  of  $Z$ . A local equation  $f$  for  $Z$  near  $p$  has the form  $f_n + f_{n+1} + \dots$ , where the  $f_i$  are homogeneous polynomials of degree  $i$  in local coordinates centred at  $p$ . The projectivised tangent cone  $\mathcal{T}_p(Z)$  to  $Z$  at  $p$  is the hypersurface in  $\mathbb{P}T_p Y$  defined by the first nonzero component  $f_n$  of  $f$ . (For a more intrinsic description, see [ACGH85, Chapter II.1].)

Now let  $C$  be a curve of genus  $g \geq (r+1)^2$ . Let  $E$  be a Petri trace injective bundle of rank  $r$  and degree  $r(g-1)$ , with  $h^0(C, E) = r+1$ . By Proposition 2.4 (with  $h = g-1$ ), the theta divisor  $\Theta_E$  is defined and contains the origin  $\mathcal{O}_C$  of  $\text{Pic}^0(C)$  with multiplicity  $h^0(C, E) = r+1$ .

By [CMTiB11, Theorem 3.4 and Remark 3.8] (see also Kempf [Kem73]), the tangent cone  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  to  $\Theta_E$  at  $\mathcal{O}_C$  is given by the determinant of an  $(r+1) \times (r+1)$  matrix  $\Lambda = (l_{ij})$  of linear forms  $l_{ij}$  on  $H^1(C, \mathcal{O}_C)$ , which is related to the multiplication map  $\bar{\mu}$  as follows: In appropriate bases  $(s_i)$  and  $(t_j)$  of  $H^0(C, E)$  and  $H^0(C, K_C \otimes E^*)$  respectively,  $\Lambda$  is given by

$$(l_{ij}) = (\bar{\mu}(s_i \otimes t_j)).$$

Hence, via Serre duality,  $\Lambda$  coincides with the cup product map

$$\cup: H^0(C, E) \otimes H^1(C, \mathcal{O}_C) \rightarrow H^1(C, E).$$

Thus the matrix  $\Lambda = (l_{ij})$  is a matrix of linear forms on the canonical space  $|K_C|^*$ .

In the following two subsections, we will show on the one hand that one can recover the bundle  $E$  from the determinantal representation of the tangent cone  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  given by the matrix  $\Lambda$ . On the other hand, up to changing bases in  $H^0(C, E)$  and  $H^1(C, E)$  there are only two determinantal representations of the tangent cone, namely  $\Lambda$  or  $\Lambda^t$ . Thus the tangent cone determines  $E$  up to an involution.

We will denote by  $\varphi$  the canonical embedding  $C \hookrightarrow |K_C|^*$ .

### 4.2 Reconstruction of the bundle from the tangent cone

As above, let  $\Lambda = (l_{ij})$  be the determinantal representation of the tangent cone given by the cup product mapping. We identify the source of  $\Lambda$  with  $H^0(C, E)$  and the target with  $H^1(C, E)$ :

$$H^0(C, E) \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{\Lambda} H^1(C, E) \otimes \mathcal{O}_{\mathbb{P}}.$$

We recall that the Serre duality isomorphism sends  $b \in H^1(C, E)$  to the linear form

$$\cdot \cup b: H^0(C, K_C \otimes E^*) \rightarrow H^1(C, K_C) = \mathbb{C}.$$

In the following proofs, we will use principal parts in order to represent cohomology classes of certain bundles. We refer to [Kem83] or [Pau03, §3.2] for the necessary background. See also Kempf and Schreyer [KS88].

**Lemma 4.1.** *Suppose that  $h^0(C, E) = r + 1$  and  $E$  and  $K_C \otimes E^*$  are globally generated. Then the rank of  $\Lambda|_C = \varphi^* \Lambda$  is equal to  $r = \text{rk}(E)$  at all points of  $C$ . In particular, the canonical curve is contained in  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$ .*

*Proof.* For each  $p \in C$ , write  $\beta_p$  for a principal part with a simple pole supported at  $p$ . Then (see [KS88]) the cohomology class  $[\beta_p]$  is identified with the image of  $p$  by  $\varphi$ . Therefore, at  $p \in C$ , the pullback  $\Lambda|_C$  is identified with the cup product map

$$[\beta_p] \otimes s \mapsto [\beta_p] \cup s.$$

The kernel of  $[\beta_p] \cup \cdot$  contains the subspace  $H^0(C, E(-p))$ , which is one-dimensional since  $E$  is globally generated and  $h^0(C, E) = r + 1$ . If  $\text{Ker}([\beta_p] \cup \cdot)$  has dimension greater than 1, then there is a section  $s' \in H^0(C, E)$  not vanishing at  $p$  such that

$$[\beta_p \cdot s'] = [\beta_p] \cup s' = 0 \in H^1(C, E).$$

By Serre duality, this means that

$$[\beta_p \cdot \langle s'(p), t(p) \rangle]$$

is zero in  $H^1(C, K_C)$  for all  $t \in H^0(C, K_C \otimes E^*)$ . Hence the values at  $p$  of all global sections of  $K_C \otimes E^*$  belong to the hyperplane in  $(K_C \otimes E^*)|_p$  defined by contraction with the nonzero vector  $s'(p) \in E|_p$ . Thus  $K_C \otimes E^*$  is not globally generated, contrary to our hypothesis.  $\square$

**Remark 4.2.** Casalaina-Martin and Teixidor i Bigas in [CMTiB11, §6] prove more generally that if  $E$  is a general vector bundle with  $h^0(C, E) > kr$ , then the  $k$ th secant variety of the canonical image  $\varphi(C)$  of  $C$  is contained in  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$ .

**Proposition 4.3.** *Let  $E$  be a Petri trace injective bundle with  $h^0(C, E) = r + 1$ , such that both  $E$  and  $K_C \otimes E^*$  are globally generated. Then the image of  $\Lambda|_C$  is isomorphic to  $T_C \otimes E$ .*

*Proof.* As  $\varphi^* \mathcal{O}_{\mathbb{P}^{g-1}}(-1) \cong T_C$ , the pullback  $\varphi^* \Lambda = \Lambda|_C$  is a map

$$\Lambda|_C: T_C \otimes H^0(C, E) \rightarrow \mathcal{O}_C \otimes H^1(C, E).$$

Write  $L := \det(E)$ , a line bundle of degree  $r(g - 1)$ . Then  $\det(K_C \otimes E^*) = K_C^r \otimes L^{-1}$ . As  $K_C \otimes E^*$  is globally generated, the evaluation sequence

$$0 \rightarrow K_C^{-r} \otimes L \rightarrow \mathcal{O}_C \otimes H^0(C, K_C \otimes E^*) \rightarrow K_C \otimes E^* \rightarrow 0$$

is exact. For each  $p \in C$ , the image of  $(K_C^{-r} \otimes L)|_p$  in  $H^0(C, K_C \otimes E^*)$  is exactly  $\mathbb{C} \cdot t_p$ , where  $t_p$  is the unique section, up to scalar, of  $K_C \otimes E^*$  vanishing at  $p$ .

Dualising, we obtain a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_C \otimes E & \longrightarrow & \mathcal{O}_C \otimes H^0(C, K_C \otimes E^*)^* & \xrightarrow{e} & K_C^r \otimes L^{-1} \longrightarrow 0 \\
& & \nearrow & & \uparrow \text{Serre} & & \\
& & T_C \otimes H^0(E) & \xrightarrow{\Lambda|_C} & \mathcal{O}_C \otimes H^1(C, E) & & 
\end{array}$$

Here  $e_p$  can be identified up to scalar with the map  $f \mapsto f(t_p)$  where  $t_p$  is as above.

Now for each  $p \in C$ , the image

$$[\beta_p] \cup H^0(C, E) \subset H^1(C, E) \cong H^0(C, K_C \otimes E^*)^*$$

annihilates  $t_p \in H^0(C, K_C \otimes E^*)$ , since the principal part  $\beta_p \cdot t_p$  is everywhere regular. Therefore,  $\Lambda|_C$  factorises via  $\text{Ker}(e) = T_C \otimes E$ . Since  $\text{rk}(\Lambda|_C) \equiv r$  by Lemma 4.1, we have  $\text{Im}(\Lambda|_C) \cong T_C \otimes E$ .  $\square$

**Remark 4.4.** A straightforward computation shows also that

$$\text{Ker}(\Lambda|_C) \cong T_C \otimes L^{-1} \text{ and } \text{Coker}(\Lambda|_C) \cong K_C^r \otimes L^{-1}.$$

We will also want to study the transpose  $\Lambda^t$ , which we will consider as a map

$$\Lambda^t: \mathcal{O}_{\mathbb{P}}(-1) \otimes H^0(C, K_C \otimes E^*) \rightarrow \mathcal{O}_{\mathbb{P}} \otimes H^1(C, K_C \otimes E^*).$$

The proof of Proposition 4.3 also shows:

**Corollary 4.5.** *Let  $E$  and  $\Lambda$  be as above. Then the image of  $\Lambda^t|_C$  is isomorphic to  $E^*$ .*

**Remark 4.6.** In order to describe the cokernel of  $\Lambda|_C$ , it is also enough to know in which points of  $C$  a row of  $\Lambda|_C$  vanishes. Dualising the sequence

$$0 \rightarrow K_C^r \otimes L^{-1} \rightarrow \mathcal{O}_C \otimes H^0(C, K_C \otimes E^*) \rightarrow K_C \otimes E^* \rightarrow 0,$$

we see that  $H^0(C, K_C \otimes E^*)^*$  is canonically identified with a subspace of  $H^0(C, K_C^r \otimes L^{-1})$ . Using the description of

$$T_C \otimes H^0(C, E) \xrightarrow{\Lambda|_C} \mathcal{O}_C \otimes H^1(C, E) \xrightarrow{\sim} \mathcal{O}_C \otimes H^0(C, K_C \otimes E^*)^*$$

as in the above proof, one sees that a row vanishes exactly in a divisor associated to  $K_C^r \otimes L^{-1}$ . Hence, the cokernel is isomorphic to  $K_C^r \otimes L^{-1}$ .

### 4.3 Uniqueness of the determinantal representation of the tangent cone

In order to show the desired uniqueness of the determinantal representation of the tangent cone, we use a classical result of Frobenius. See [Fro97] and also for a modern proof [Die49], [Wat87] and the references there. For the sake of completeness we will give a sketch of a proof following Frobenius.

**Proposition 4.7.** *Suppose  $r \geq 1$ . Let  $A$  and  $B$  be  $(r+1) \times (r+1)$  matrices of independent linear forms, such that the entries of  $A$  are linear combinations of the entries of  $B$  and  $\det(A) = c \cdot \det(B)$  for a nonzero constant  $c \in \mathbb{C}$ . Then, there exist invertible matrices  $S, T \in \text{Gl}(r+1, \mathbb{C})$ , unique up to scalar, such that  $A = S \cdot B \cdot T$  or  $A = S \cdot B^t \cdot T$ .*

*Proof by Frobenius [Fro97, pages 1011-1013].* Note that for  $r \geq 1$  only one of the above cases can occur and the matrices  $S$  and  $T$  are unique up to scalar. Indeed, let  $A = SBT = S'BT'$  and set  $b_{ii} = 1$  and  $b_{ij} = 0$  if  $i \neq j$ , then  $ST = S'T'$ . Set  $U = T(T'^{-1}) = S(S'^{-1})$ , thus  $UB = BU$ . Since  $U$  commutes with every matrix, we have  $U = k \cdot E_r$  and hence  $S' = k \cdot S$  and  $T' = \frac{1}{k} \cdot T$ . Similar one can show that  $B^t$  is not equivalent to  $B$ . Note also that there is no relation between any minors of  $A$  or  $B$ .

For  $l = 0, \dots, r$ , let  $c_{ij}^l$  be the coefficient of  $b_{il}$  in  $a_{ij}$  and let  $y$  be a new variable. We substitute  $b_{il}$  with  $b_{il} + y$  in  $A$  and  $B$  and get new matrices, denoted by  $(a_{ij} + y \cdot c_{ij}^l)$  and  $B^l$ , respectively. Since  $\det B^l$  is linear in  $y$ , the coefficient of  $y^2$  in  $\det(a_{ij} + y \cdot c_{ij}^l) = \det B^l$  has to vanish. But the coefficient is the sum of products of  $2 \times 2$  minors of  $(c_{ij}^l)$  and  $(r-1) \times (r-1)$  minors of  $A$ . Since there are no relations between any minors of  $A$ , all  $2 \times 2$  minors of  $(c_{ij}^l)$  vanish. Hence,  $(c_{ij}^l)$  has rank one for any  $l$  and we can write  $c_{ij}^l = p_i^l q_j^l$  where  $p^l$  and  $q^l$  are column and row vectors, respectively.

Let  $B_0 = B|_{\{b_{ij}=0, i \neq j\}}$  and  $A_0 = A|_{\{b_{ij}=0, i \neq j\}}$ . Then

$$A_0 = PB_0Q$$

where  $P = (p_i^l)_{0 \leq i, l \leq r}$  and  $Q = (q_j^l)_{0 \leq l, j \leq r}$ . Since  $\det(A_0) = c \cdot \det(B_0) = c \cdot b_{00} \cdot \dots \cdot b_{rr}$ , we get  $\det(P) \cdot \det(Q) = c$ , hence  $P$  and  $Q$  are invertible.

Let  $\tilde{B} = P^{-1}AQ^{-1}$ . By definition  $\tilde{B}|_{\{b_{ij}=0, i \neq j\}} = B_0$ . Thus, the entries  $\tilde{b}_{ij}$  for  $i \neq j$  and  $v_i = \tilde{b}_{ii} - b_{ii}$  are linear function in  $b_{ij}$  for  $i \neq j$ . Furthermore, we have

$$\det(\tilde{B}) = \det(P^{-1}AQ^{-1}) = \det(P^{-1}Q^{-1}) \cdot \det(A) = \frac{1}{c} \cdot c \cdot \det(B) = \det(B).$$

Comparing the coefficient of  $b_{11}b_{22} \cdots b_{rr}$  in  $\det(\tilde{B})$  and  $\det(B)$ , we get  $v_0 = 0$ . Similarly,  $v_i = 0$  for  $0 \leq i \leq r$ . Comparing the coefficients of  $b_{22} \cdots b_{rr}$ , we get  $b_{12}b_{21} =$

$\widetilde{b}_{12}\widetilde{b}_{21}$  and in general

$$b_{ij}b_{ji} = \widetilde{b}_{ij}\widetilde{b}_{ji}, \quad i \neq j.$$

Comparing the coefficients of  $b_{33}\cdots b_{rr}$ , we get  $b_{12}b_{23}b_{31} + b_{21}b_{13}b_{32} = \widetilde{b}_{12}\widetilde{b}_{23}\widetilde{b}_{31} + \widetilde{b}_{21}\widetilde{b}_{13}\widetilde{b}_{32}$  and in general

$$b_{ij}b_{jk}b_{ki} + b_{ji}b_{ik}b_{kj} = \widetilde{b}_{ij}\widetilde{b}_{jk}\widetilde{b}_{ki} + \widetilde{b}_{ji}\widetilde{b}_{ik}\widetilde{b}_{kj}, \quad i \neq j \neq k \neq i.$$

A careful study of these equations shows that either

$$\widetilde{b}_{ij} = \frac{k_i}{k_j}b_{ij} \text{ and } \widetilde{B} = KBK^{-1} \text{ or } \widetilde{b}_{ij} = \frac{k_i}{k_j}b_{ji} \text{ and } \widetilde{B} = KB^tK^{-1}$$

where  $K = (k_i\delta_{ij})_{0 \leq i, j \leq r}$ . The claim follows.  $\square$

Let  $\Lambda = (l_{ij})$  be a determinantal representation of the tangent cone  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  as above. Since the bundle  $E$  is Petri trace injective, the matrix  $\Lambda$  is  $(r+1)$ -generic (see [Eis88] for a definition), that is, there are no relations between the entries  $l_{ij}$  or any subminors of  $\Lambda$ .

**Corollary 4.8.** *For a curve of genus  $g \geq (r+1)^2$  and a Petri trace injective bundle  $E$  with  $r+1$  global sections of degree  $r(g-1)$ , any determinantal representation of the tangent cone  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E) \subset |\mathbf{K}_C|^*$  is equivalent to  $\Lambda$  or  $\Lambda^t$ .*

*Proof.* Let  $\alpha$  be any determinantal representation of the tangent cone  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  in  $|\mathbf{K}_C|^*$ . Then,  $\alpha$  is an  $(r+1) \times (r+1)$  matrix of linear entries, since the degree of the tangent cone is  $r+1$ . Furthermore, the entries  $\alpha_{ij}$  of  $\alpha$  are linear combinations of the entries  $l_{ij}$  of  $\Lambda$ . Indeed, assume for some  $k, l$  that  $\alpha_{kl}$  is not a linear combination of the  $l_{ij}$ . Then,  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  would be the cone over  $V(\alpha_{kl}) \cap \mathcal{T}_{\mathcal{O}_C}(\Theta_E)$ . Hence, the vertex of  $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$  defined by the entries  $l_{ij}$  would have codimension strictly less than  $(r+1)^2$ ; a contradiction to the independence of the  $l_{ij}$ . The corollary follows from Proposition 4.7.  $\square$

## 5 Injectivity of the theta map

**Theorem 5.1.** *Suppose  $r \geq 3$ . Let  $C$  be a Petri general curve of genus  $g > (r+1)^2$ . Then the theta map  $\mathcal{D}: SU_C(r) \dashrightarrow |r\Theta|$  is generically injective.*

*Proof.* Firstly, note that the Brill-Noether locus  $B_{r, r(g-1)}^{r+1}$  consists precisely of those bundles of the form  $V \otimes M$  where  $V \in SU_C(r)$  and  $M \in \text{Pic}^{g-1}(C)$  and  $h^0(C, V \otimes M) \geq r+1$ . Since  $C$  is Petri general and  $g \geq (r+1)^2$ , by [BBPN08, Theorem 3.1 and Remark 6.2] the locus  $B_{r, r(g-1)}^{r+1}$  is irreducible and a generic element  $E$  is generated by global sections. It follows that in general  $\mathbf{K}_C \otimes E^*$  is also globally generated. Furthermore,

by Lemma 2.5 and openness of Petri trace injectivity, a general such  $E$  is Petri trace injective and satisfies  $h^0(C, E) = r + 1$ .

Let  $V \in SU_C(r)$  be a general stable bundle. By Proposition 2.6, we may choose a point  $M$  such that  $h^0(C, V \otimes M) = r + 1$  and furthermore  $V \otimes M =: E$  is Petri trace injective. By Proposition 2.4, the theta divisor  $\Theta_V$  exists and has multiplicity  $r + 1$  at  $M$ . By the first paragraph,  $E$  and  $K_C \otimes E^*$  may be assumed to be globally generated.

Note that tensor product by  $M^{-1}$  defines an isomorphism  $\Theta_V \xrightarrow{\sim} \Theta_E$  inducing an isomorphism  $\mathcal{T}_M(\Theta_V) \xrightarrow{\sim} \mathcal{T}_{\Theta_C}(\Theta_E)$ . In order to use the results of the previous sections, we will work with  $\Theta_E$ .

Now let

$$\alpha: \mathcal{O}_{\mathbb{P}^{g-1}}(-1) \otimes \mathbb{C}^{r+1} \rightarrow \mathcal{O}_{\mathbb{P}^{g-1}} \otimes \mathbb{C}^{r+1}$$

be a map of bundles of rank  $r + 1$  over  $\mathbb{P}^{g-1}$  whose determinant defines the tangent cone  $\mathcal{T}_{\Theta_C}(\Theta_E)$ . By Corollary 4.8, the map  $\alpha$  is equivalent either to  $\Lambda$  or  $\Lambda^t$ , where  $\Lambda$  is the representation given by the cup product mapping as defined in §4. Therefore, by Proposition 4.3 and Corollary 4.5, the image  $E'$  of  $\alpha|_C$  is isomorphic either to  $T_C \otimes E = V \otimes M \otimes T_C$  or to  $E^* = V^* \otimes M^{-1}$ . Thus  $V$  is isomorphic either to

$$E' \otimes K_C \otimes M^{-1} \quad \text{or to} \quad (E')^* \otimes M^{-1}. \quad (9)$$

Now since we are assuming strict inequality  $g > (r + 1)^2$ , by Remark 2.7 the open subset  $\{M \in \text{Pic}^{g-1}(C) : h^0(C, V \otimes M) = r + 1\} \subseteq B_{1, g-1}^{r+1}(V)$  has a component of dimension  $g - (r + 1)^2 \geq 1$ . Therefore, we may assume that  $M^{2r} \not\cong K_C^r$ . Thus only one of the bundles in (9) can have trivial determinant. Hence there is only one possibility for  $V$ .

In summary, the data of the tangent cone  $\mathcal{T}_M(\Theta_V)$  and the point  $M$ , together with the property  $\det(V) = \mathcal{O}_C$ , determine the bundle  $V$  up to isomorphism. In particular,  $\Theta_V$  determines  $V$ .  $\square$

**Remark 5.2.** The involution  $M \mapsto K_C \otimes M^{-1}$  defines an isomorphism of varieties  $\Theta_V \xrightarrow{\sim} \Theta_{V^*}$ . We observe that the transposed map  $\Lambda^t$  occurs naturally as the cup product map defining the tangent cone  $\mathcal{T}_{K_C \otimes M^{-1}}(\Theta_{V^*})$ .

**Remark 5.3.** If  $C$  is hyperelliptic, then the canonical map factorises via the hyperelliptic involution  $\iota$ . Thus the construction in §4.3 can never give bundles over  $C$  which are not  $\iota$ -invariant. We note that Beauville [Bea88] showed that in rank 2, if  $C$  is hyperelliptic then the bundles  $V$  and  $\iota^*V$  have the same theta divisor.

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